

Imagination and Mathematics: Geometry of Thought

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The Philoctetes Center

Levy: Francis Levy
Nersessian: Edward Nersessian
Mazur: Barry Mazur
Brann: Eva Brann
Audience: Question from audience

Levy: I'm Francis Levy, co-director of The Philoctetes Center. Dr. Edward Nersessian is the other co-director, and welcome to *Mathematics and Imagination: Geometry of Thought*, a much anticipated roundtable which has resulted in a great deal of discussion. I am pleased to introduce Barry Mazur. Barry Mazur is Gerhard Gade University Professor of Mathematics at Harvard University. He does research in number theory and geometry and is the author of *Imagining Numbers: (particularly the square root of minus fifteen)*. Dr. Mazur will lead tonight's discussion and introduce our distinguished guests. Thank you.

Mazur: Well then I have the honor of introducing Eva Brann. Eva has been teaching at St. Johns College in Annapolis for half a century, is the recipient of quite a number of awards including the Presidential Medal for the Humanities, and I didn't realize that I was supposed to introduce Eva, otherwise I would have written a half-hour list of her accomplishments, but one of them is a marvelous book called *The World of the Imagination: Sum and Substance*. Eva has taught mathematics and many other things for a long time and is very, very interested in the interplay between the imagination and mathematics. Is that reasonable?

Brann: That's very reasonable.

Mazur: Okay. The first thing I want to say is that I don't think you've had a roundtable or a course in mathematics before.

Levy: No we haven't.

Mazur: You haven't. So therefore the format is up for grabs. That is to say, we can decide how we're going to do it. And the only way to do it so that everybody is engaged is to feel free to interrupt, ask a question, ask us to repeat something if we haven't, or—

Brann: Give us an argument.

Mazur: Give us an argument, then make a comment, okay? The first thing that I want to say, given the fact that I don't know anybody's background in the audience, is that mathematics is in the air. No matter what, it has a bum rap in the sense that it's considered an obscure subject. It seems to require people say years of devotion and learning and even this funny, funny word, aptitude. But what I want to do is make an end run around all this and first convince you that no matter what your background is you probably have not only absorbed a lot of mathematics, but

you may have absorbed some of the deepest mathematics. Here is an example, we say, far in the future, and we're happy to say that. And what have we just done when we've said this? We have made the metaphor of time as being geometric. The metaphor of time as geometry. Now once we do this we have a completely different vocabulary for time.

If you've ever seen, and I'm sure everybody has seen graphs, graphs are part of the air we breathe. Stock market graphs, these are sort of depictions of time where when we talk about them we don't use necessarily purely temporal vocabulary. We have, so to speak, recruited by this metaphor, time as on a line, time as geometric. We've recruited an immense amount of vocabulary that would otherwise be unavailable to us. We talk about the peaks and the valleys in the stock market, we talk about de-sets and assets, and we're perfectly happy to do this.

On the other hand, we might go the other way. We might refer to distance in terms of time as light-years or the old fashioned "two cigarettes later," that sort of thing. We are perfectly happy to metaphorize one quantity, a geometric quantity for an algebraic quantity, or the other way around. We can turn geometry happily into algebra when we say 8th Avenue and 42nd Street. We're talking all numbers, but we're specifically thinking of a spot in New York City. So our ability to turn algebra into geometry is part of the air we breathe. Our ability to metaphorize things: time, temperature, humidity, as geometric quantities is something that we all do naturally, and this is a long tradition.

There is a 14th century **bishop of Lisieux, Nicole Oresme, who** had a treatise that he called the "Geometry of Qualities." He is the first guy who invented graphs. He would see, say, the quality of motion as going in an ascending line, and he would think of that as a way of depicting acceleration. We also talk about speed without blinking an eyelash, and that is an incredible mathematical metaphor. All of these things that we do naturally have the effect of recruiting a vocabulary that has its root or its anchor in one intuition, geometry, let's say, and applying it to the names that we would otherwise not think of as geometric. Or recruiting algebra and applying in the names that we would not think of as algebraic. Let's call it the metaphors of recruiting one vocabulary anchored in one intuition—the intuition say of vision or geometry—to another domain, or the reverse: the vocabulary of numbers to a domain that otherwise would not be considered numbered.

We've all done this, and these are some of the biggest leaps of the imagination. I just want to open the discussion this way to suggest that when we talk, if we do get to talk about other directions, other leaps of the imagination, everyone in this room should understand that we've been doing this for a long time and we do it almost without effort. Okay, this is a spiel. Do I have any comments first on that?

Audience: I just want to add to that you do that in gestures too. I was watching your hands. When you talked about time you made a movement from left to right which is characteristic of writers of Western language: to think of time as moving from left to right and make that gesture when they talk about time.

Mazur: Okay, in fact what you've done is you've reminded us that the body is absolutely essential for most of the metaphors that we're most comfortable with. That's part of it.

Audience: Sure, that's another way of—communication is quite visual in the way that we use our hands.

Mazur: Any other comments?

Audience: I have an idea: how far an equation—

Mazur: An equation. Yes?

Audience: Would be equal to or beyond a metaphor.

Mazur: Beyond a metaphor.

Audience: A metaphor. Especially when you talk about dreams for example.

Mazur: Dreams, yes. Okay, so, let me take the first part of your question about equations as metaphors. You know 'x' equals 'y' and you might say 'x' equals 'y' or is metaphorically 'y'. That might be the drift of what you're saying. I would say it's also the source of really fascinating metaphors. For example, there are standard equations of hydraulics where the vocabulary is water pressure, the size of pipes, the speed of water. Water is an incompressible fluid so you have very precise equations which describe that sort of thing. Now these equations are so useful that you can't just let them live in one domain only. So when people began to think of electricity they tried to recreate exactly the same equations, if you wish, but for electricity. Now for that you have to talk about pressure. You have to talk about resistance. There may be a sort of friction, that kind of resistance. You have to talk about various other things like kinetic energy as well, but in the language of electricity, and one of the templates that early electrical equations had were hydraulic metaphorical templates. Well that's electricity, electricity going through wires.

Maxwell, who discovered that when something happens electrically that doesn't involve wires—in other words, radio—realized, (I guess others too realized) that there is a problem. If you're going to conserve energy and there's basically radio waves, how is it that radio waves, which travel at finite distance from the transmitter to the receiver—this is totally anachronistic vocabulary if we're going to talk about Maxwell, but anyway. The radio waves move at finite velocity, so if you don't take account of those radio waves while they're neither in the transmitter or in your receiver, but somewhere in the air, you lose conservation of energy. The only way to take account of it is to maintain stubbornly and dogmatically exactly the same equations that started out in hydraulics and then had their continued life in the electricity—you know, sort of simple electrical circuits—and now had to account for the fact that a certain amount of the energy was moving through a domain that was not wire and therefore was not part of the structure that the equations were dealing with.

What he did was he actually used exactly the same letter even, 'j', for the current that would go through a wire—which itself was metaphorically water going through a pipe—for electromagnetic waves going through space, so as to retrieve, or at least sort of somehow keep up, the structure of the equation in this new metaphorical realm. So what you're saying, I think,

is absolutely right. I mean the equations are not only themselves metaphors, but they're the seeds of lots of metaphors. Wouldn't you say?

Brann: I heard two questions. The first one is, isn't it the case that an equation has two aspects. One is what goes into it: numbers or physical quantities, constance, variables. The other one comes much earlier and it seems to me is even more metaphorical, namely that it's a kind of balanced lever, so that what you do to one side you have to do to the other and if you want to shove everything on one side over on the other side you've got to do that in a certain way, such that the balance is preserved. Now it seems to me this is a really interesting metaphor and I think it took a long time for people to recognize that one could do that, in other words that one could construct something that had all the characteristics of a balance, and then one had to obey rules that kept those characteristics. I think that comes even before the actual matter. The other question is whether people agree that that's an even more fundamental characteristic of an equation. The other question I had was you were talking about the metaphorical transfer of meanings. What is it that the non-metaphorical elements, symbols, are about? When you have the physical interpretation it could be hydraulics or it could be waves. But what is it about when there are no physical interpretations?

Audience: It's one of the questions that I'm asking, because taking a step back before you get to the metaphor, what is mathematics? A sentence, a construct of the human mind, the language in which reality is written or—.

Mazur: Okay, let's get to that. There was another question back there first.

Audience: Essentially, it was replicating the same idea, which is that are we presuming—and I got this feeling when you were talking about mathematical metaphors and using those, so when we say, 'time is moving forward,' we're having access to a sort of mathematical concept. But I can't help but wonder, are we assuming that the universe does map, and that we are then sort of describing it, or accessing this map that's being done through our sort of metaphors and hence accessing math through our metaphors, or are we—because the primary way that I have been approaching math recently is that it is the closest approximation to predict how reality will behave in the future. So both are sort of competing—math as metaphor.

Mazur: That's a version of your question.

Brann: I want to say something about that particular metaphor that is being far in the future. It seems to me the most peculiar metaphor in the world because not only does it spacialize, it specializes something that is not a space and that doesn't exist.

Mazur: What's that?

Brann: The future.

Mazur: Oh, the future. Yeah. We'll get to that. Go ahead, ask your question, and we'll get to that.

Audience: I want to point out this connection that cosmologists like Brian Greene are always interested in, the question of why mathematics seems to describe the universe so effectively.

They ask why. What is it about math that should be able to do this? If at some point in the evening Dr. Brann maybe you can get a chance to just talk about that a little I think it would be very interesting.

Mazur: Well, there are various possibilities so we can vote on this at the end. I'll just give you a few of them. Here's one: mathematics is in fact the way we construct our presentations of the world. It's the fundamental vocabulary of the way in which we piece things together. If it's not amenable to that vocabulary, we'll never piece it together. Everything that seems to be such a beautiful fit is because it's almost the anthropic fallacy—because that's how we think, and that's what we mean by explanation in the end. Anything that is amenable to this works. It's sort of like Freud at one point—I think it's Freud. Uh-oh.

Nersessian: I am not the Freud police.

Mazur: All right. You correct me whether Freud was the person who said, "Isn't it amazing that the cat's fur knows where to put two holes exactly where its eyes are." I mean, we have a certain amount of a repertoire of explaining power.

Brann: Barry, I've got to interrupt here. That doesn't explain enough. I thought the question was how does mathematics happen to fit the world? The fact that we think mathematically wouldn't be good enough if the world wasn't amenable to being thought about mathematically.

Mazur: How about, those parts of the world that are not amenable to being thought about mathematically will escape us?

Brann: Yeah, that may well be. It's still a question of what it is about the world that—in some ways it's obvious. For instance everybody here is a unit and that adds up to number. But in other ways, it's not so obvious.

Audience: But you see you're completely reflecting the two different points of view. One is that it's a constructed—

Mazur: Oh yeah, I'm not finished.

Audience: --And one is this is a reflection of reality.

Brann: Right.

Mazur: We have to vote, because I just gave you one. There's the absolute opposite view, which is that what we think of the mathematics that we are constructing is really the architecture of the cosmos that we're observing. We observe this and we put it down in sort of a reasonable form and of course it fits because that is the faithfulness of our observation. That's the Platonic view: that we are observers of architecture when we're mathematicians. When we carefully set down our mathematics, what we're doing is we're carefully setting down elements of the structure of the cosmos. In the end we take a look at it and we say, my god, it fits so well. Well it fits because we are good observers. That's the Platonic. The Kantian is the first thing that we could vote on. There are others, and I think it's much more interesting to ask the question and to weigh these

possibilities—pure Platonic against the Kantian—and see to what extent does it ring true, to what extent can one make sense of it, and to what extent does it lead you to further insights?

Audience: In your first explanation it can't be right that mathematics is the language of explanation because you're talking words, not mathematics. You're explaining things in words and that's just fine. I think if you explained it in mathematics we wouldn't understand. When I watch mathematicians talking to each other about mathematics they use words too, in fact they use metaphors, mixing them. I wanted to sort of propose a third way, that might be that we make the mathematics fit what we think the world is like. Mathematics isn't a single thing. There are many kinds of mathematics, many kinds of feelings that can be incorporated in math, and we pick the one that seems to fit the world, which doesn't feel exactly put upon. It feels a little different. I don't know whether either of you would agree with that.

Brann: I think we're mixing apples and oranges here. I don't know which is the apples and which is the oranges, but one of these fruits is the question of why mathematics fit the world. In other words, why we can attach a mathematical formula to an event in the world be it free fall, be it a situation of attraction, whatever the case may be. In a very simple sense, why can mathematics describe motions in the world?

The other question seems to me—I don't think they're the same—where the mathematics that is pure and unapplied, that is the non-physical mathematics, comes from. Do we get that from the world? Barry and I were talking about Hume before. Hume is actually convinced that we get mathematics by observing the world. It seems like a kind of madness to me. It doesn't come from there. Aren't these different questions?

Mazur: I think they're different questions, but I like the English. I like it very much. That's the third one that we should vote on. That in fact, everything is language, our native language, and what we're doing is we're building on our native language in various directions. One way of building, given the metaphors of space and number and the various other metaphors that are sort of the building blocks of mathematics, one way of working on language so as to have enormous power is the mathematical way. Maybe what you're saying is we shouldn't distinguish this from other ways of working on language so as to have enormous power. The root there is language, I mean, and then you know—when Socrates asked the slave boy to—let's actually do a mathematical problem—he didn't know what his background was in mathematics. He just asked, "Do you know Greek?" and the person said, "yes," and then he repeats it, "Do you know Greek?" He says, "yes," and then he's off. So it's possible that language is the root of everything and that there are certain off-shoots of language which require, again a sort of a focus on metaphor, and correspond to a lot of things, including mathematics.

Brann: Yeah, but. I keep saying but.

Mazur: But. Go ahead.

Brann: It's true he's talking language all the time. He usually does it. But it's also the case with the example that he mentioned, where they're trying to find what the relationship of a diagonal to a side of a square is. It ends in what is, I imagine, the most important set of numbers in our world, irrationals. That is to say unspeakables. That's what the word means.

Mazur: Oh.

Brann: Yes, say something.

Mazur: Okay. There are lots of questions. Happily, I don't have to answer that one.

Audience: I think something that might be an 800-pound gorilla in the room, which you suggest by the point of language and the point of metaphor and the point of talking about systems, is that mathematics is, in some ways, describable as a system and we know through **Gortel** that the system is limited. Yet somehow through the paradoxes that we derive from incompleteness and the inability of a system to be so robust as to prove every statement in its own set true that there's something about our abilities to understand phenomenal experience that allows us to appreciate the paradox without being able to explain it or to be able to solve why there is this limitation. This to me roots back in the point that you made about thought and about the fact that as humans with the perceptual apparatuses and conceptual processes, we are dealing with phenomena in the world which we have a kind of access to and all of our processing seems to be predicated upon the kind of access we have to phenomenon, whether they're in the world or in our heads. There's still this fundamental set of operations we are doing and there might well be limitations to that set of operations that we can appreciate without fully exploiting them. It's related to your question.

Mazur: The minute that you have a vocabulary which you put in a system, if the system is sufficiently rich and reasonable there will be questions that lead outside the system, that you can pose within the system but you can't answer within the system. That's **Gortel's** discovery. The minute you fix on a system there will be something in that system that will break out. With any such numerical issue, for example, you have the square whose sides are one and one and then you draw the diagonal, the first interesting line you might draw in Euclid, and you ask what is the ratio of the diagonal to the edge of the square, and you discover that if you are within the system of fractions, of rational numbers, you will never be able to answer that question explicitly in the terms of the system. So you break out of that system and you say it's irrational but it's irrational relative to that system, but of course you build a bigger system where it's perfectly rational. Although not by name because you've called it irrational.

Brann: And because you don't have words for it, you have a name for it.

Mazur: No, I have words for it and I have a name for it. It's just not rational.

Audience: Well I think we have to appreciate the order of the phenomena.

Mazur: Yes.

Audience: Just some simple facts: the universe as we've calculated is fifteen billion years old, the earth is four billion, and interestingly enough the existence of the logic of the natural phenomena pre-dated the development of mathematics by billions of years.

Mazur: The logic of the processes, in other words, pre-dated our understanding of math.

Audience: Right, which leads to the conclusion that nature has been essentially dominated by a sense of relationships that have a certain regularity. When you look at mathematics and made it come into existence in its most permanent form 5,000 years ago, this was in some ways an effort to understand the existence of rules and relationships, and therefore we can say that mathematics is understandably related to the universe because it's partly in comprehension that underlined the way in which nature created things. On one hand we can feel very fortunate that there is a regularity in this world, that we do not have a kind of supernatural phenomenon where we're constantly confronted with the fact that every relationship between certain factors is completely irregular.

Mazur: Right. I agree with you, but can I label what you say the efficacy of mathematical physics? But there is mathematics that has nothing to do with the natural world, and that's the type of mathematics that Eva was talking about. The type of mathematics that asks is the ratio of the diagonal—the length of a diagonal of a square to its side—is that expressible in whole numbers? That has nothing to do with the natural world or pi. You could have either the area of a circle, of a given radius, or the circumference of a circle of a given radius, and you ask what is the relationship of that number—either of those two numbers—to the radius. This on the one hand is a very unnatural question to ask because we can approximate it. We can build wheels. We can build any approximation we want, but when we ask the mathematical question, what is pi, we are going beyond any approximation. It only becomes interesting if we make no approximation—or there are other interesting questions to ask—but the primary interesting question about the square root of two or pi: these are highly unnatural questions. What's baffling is how persistent these questions are and how uniform a structure of explanations we have to encompass these questions, but equally we have a uniform structure to encompass the questions of the natural world.

Audience: What I'm not all right with is this. The universe doesn't have to absorb every mathematical formulation, but the fact is that the universe, in so far as we know it now, has fairly accurately described the various mathematical formulations. The further development of mathematics is still in the abstract and whatever practical utility may be further discovered, but we can also appreciate it as just a mere development of a certain logic, mathematical logic.

Audience: How can you integrate quantum mechanics into that?

Audience: Quantum mechanics satisfiably resolves sub-atomic phenomenon in a very adequate way with predictable equations and there's no difficulty. The problem is integrating quantum mechanics with relativity.

Mazur: Okay, now these are very lofty issues. There is relativity, quantum mechanics, and of course the full panoply of physical laws regarding the natural world, billions of years old. Let's go back to more primitive issues that we can personally attest to, that we can have our own personal autonomy about, for example, the simplest constructions in geometry. Now if you think of it, any time you have a question in geometry that requires a construction, the question never asks of the construction. The question asks something else. The construction comes as, let's say, a bolt out of the wilderness to change your own conception of the issues. I have prepared something here which is a very simple diagram. I drew it, so it's not perfect, but it's two

parallelograms and it's meant to be two parallelograms of the same base length and the same height but of different, let's say different lean-tos, so to speak.

Now it's a fact, due probably to people before Euclid, that the area of these two parallelograms are the same. In fact, the area of any two parallelograms that have the same base length and the same height, these areas are the same. How do you wrap your mind around this? How would you go about showing such a thing? But of course if you know the answer, you know the answer. If you don't know the answer, then it's interesting because you have to think, my gosh, this guy is showing me two parallelograms. They don't look like they have the same area. They don't even look like they have the same base length and height—but I measured them and they do, at least to the width of my pen—and so therefore they have the same area but they don't look like they have the same area. How would I ever do such a thing as to convince myself of this? Usually if you ask yourself there is the big question that you will not have asked, which is, what is area?

Well, here's the construction. You take the parallelogram and you draw a perpendicular to one of its sides. What have you done? You've broken the parallelogram into some funny figure plus a triangle. Now you just move that triangle so as to replace the parallelogram by the rectangle of exactly the same base length and of exactly the same height. So, what has the construction done for you? It has said that there is a process: cutting and moving. You have to make a construction, you construct a line, you cut, and you move the triangle over and you've regularized every such parallelogram and you've made it out of the same material, so to speak, as the rectangle, so of course it has the same area. When I say of course it has the same area, I've snuck into the discussion the definition of area. I haven't quite defined it, but clearly if I cut a figure into little pieces and move the pieces around and make another figure those two figures are going to have the same area. Any construction like this has the effect of completely reorganizing the question it purports to answer. One of the problems in teaching mathematics is that once one does such a construction and goes onto the next thing—whereas one could spend two months on just this construction, because there's a lot to understand about it.

Audience: When I do that with my ten-year-olds and we cut them up and move them around, then they have no problem with the formula and they love it and everything, but the thing that they don't understand is why I get so bent out of shape if they don't put square units in their answer. I keep trying to explain that it's a whole surface, it's a two-dimensional thing. They can figure out that I get bent out of shape, but that's the thing they don't really get. Why the square units? What's that about?

Mazur: That's the stealth issue in this whole thing. It's just out of drama. You don't define area to do this, because if you did, you'd be spending a week defining area. What you're doing is you're setting up the situation where it will eventually become absolutely clear to the student how to define area, given that this is the process that identifies two figures as having the same area.

Brann: But you've run an end-run around the question of the definition of area.

Mazur: Yes.

Brann: Instead you've shown what equal areas are.

Mazur: Exactly.

Brann: Indecently, if it becomes clear what an area is, what is it?

Mazur: What is an area?

Brann: Yeah.

Mazur: Well, here's a thought. You give me your most favorite figure.

Brann: As you know, that's my most favorite figure.

Mazur: The rectangle?

Brann: The one—well, the way I think of it is it's between rails, right? You can take it and pull it as far as you like, it's thinner and thinner and thinner. I don't know if your students do that. It's a lot of fun to do. You just take the rectangle or parallelogram in the area, you keep the base fixed, you take the top, which is equal to the base, you start pulling it along. And you pull and pull and pull and the thing gets thinner and longer and longer. And my guess is—weren't you telling me about someone who studies how people are fooled by mathematical, this business about 8, 7—

Mazur: Oh, okay, yes. I'll tell you about that.

Brann: Tell us about that.

Mazur: No, but go ahead. Continue.

Brann: I was going to say, this seems to me a similar case. People think that because something is long, long, long, it's got to be larger than something that is short, short, short. I love that. It's beautiful.

Mazur: You're right, when I made an end-run around the definition of area which is—

Brann: But explain the—I mean just because it's nice.

Mazur: Okay, well this is in the context of our intuition sometimes fools us. Alfred Tarski was one of the great scientists who developed a number of, let's call them experiments, and they're experiments he did with a number of subjects—fifty subjects, let's say, but they're very personable experiments. You could do it to yourself and you're as convinced if you do it to yourself that there's something strange about your intuition. In the case of elongated figures having larger area, that's one, but the other is in numbers. Here's a test which I'm going to ask you to make a guess, but the guess can be totally private so there's no shame, and don't take out your calculators. You have to guess. What is this number? Guess the number: $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$? Okay? Secret guesses, but you have to have a number.

Everybody make the guess. And now if you've made the guess and it's wrong, you're not alone, by the way. Tarski would ask a certain number of subjects, let's say fifty subjects, and he would tabulate their guesses and there's an average guess for this experiment. I'll call that average

guess X . I don't remember what it is. Then he would take fifty other subjects and ask them a completely different question: $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10$. All fifty subjects would write down their answers and there would be another answer average, sort of a mean, Y . Y is significantly smaller than X . By the way, X is significantly smaller than the true answer, which is 3.5 million, roughly.

Alfred Tarski had a category for this, he called this "framing," I believe. That is to say, when we're asked a question, $10 \times 9 \times \dots$ —we're thinking big number, big number, big number, and then it's going down, and even though we could rationally sort of re-group and reassess at the end of the description of the number exactly how big a number it is, the fact that our first impression of it was big, $10 \times 9 \times 8 \times \dots$ so on, influences our sort of rapid estimate of the number. If we started $1 \times 2 \times 3$, well we'd think, small number, small number, small number, and that influences the outcome as well. So one of the things that I guess we began this discussion with, the intuitions of mathematics, reside to a large extent in the body, or at least are related to aspects of our physical being, of our physical intellectual being, so to speak. Here's another example where we have a kind of hard wiring predilection to favor the beginning impressions of something even if it is as abstract and as objective as a number. So that's what Eva wanted me to recall.

Brann: Back to area.

Audience: Suppose I call the number n factorial. Do I know that number? I mean, is just calling it n factorial and calling it 3.5 billion—is the 3.5 billion more knowable than the n factorial?

Mazur: Is it more what?

Audience: If I just give it a name do I know that number? Why does 3.5 billion make it more knowable?

Mazur: Yeah, that's a good question. The thing is, what is knowable depends upon its own prior vocabulary. We very often are much more comfortable with decimal notation than we would be with, for example, binary notation. If I gave you the answer in binary you might say, do I know this or what? And why is that any different from just saying it's 10 factorial? On the other hand, to compare two numbers some systems are better than others and so I was interested in the comparison and the size of it. If I said to you, two to the tenth minus one, or something, you would have difficulty comparing it something like three to the eighth plus five. More difficulty than if I put it in decimal notation. So every way of expressing something in vocabulary puts a spotlight on some things and allows you to do certain things easily, other things less easily.

Brann: Right. In this case it seems to me the reason that 3.5 million is somehow more substantial than 10 factorial is that 10 factorial is an invitation to do an operation and 3.5 million is an outcome, and I like outcomes better than I like operations.

Mazur: Don't I have to multiply a million by 3.5?

Audience: Also numbers are related to fingers. You know, in your earliest stage you relate to digits.

Mazur: Right, sure.

Audience: I find that most people cannot comprehend the size of large numbers and I think that it explains a lot that goes on in politics. A nice question to ask people is how long they think a billion seconds is and have people make a guess of that, and then if one's teaching children you might have them actually do the calculation. It helps one realize how much we can't really comprehend what a billion means much more than—

Mazur: That goes back to your question, that maybe “10 factorial” for certain people would be more telling than 3.5 million. We have a difficulty really coming to grips with large numbers. Absolutely.

Audience: It seems we have an impulse to grasp something concretely, in other words as actualities as opposed to realities, which are accounts of things. If we're talking about area, it's fairly easy to say what the area is, but it's very, very difficult, probably impossible, to know what area is because that's the concrete question.

Mazur: Okay, Francis. Yes?

Levy: This is going to sound kind of stupid, but when you showed that diagram, I was trying to figure out how I would compose this, so I said to myself angulation, that's it. I think that's what's wrong, that one tries to find one variable. And I thought, well, the sun rises at two angles. I immediately tried to figure something out, but I was limiting the possibility in order to comprehend what you were trying to get at.

Mazur: Well what limits—first, that's true. Secondly it's inevitable, and sometimes there's strength in living your possibility for a short time, so to speak, and being willing to change if it doesn't work. I have an example but there's another question which I'll get to later.

Audience: It's not a question really, but in thinking about what is area, is that any different than what is three? There's a three-ness about stuff and there's an area about stuff.

Mazur: Well, the thing is, if someone sort of held me here and said, “Define area,” I know how I would do it. But if they then say, “What *is* area,” after I define area, that's a philosophical issue which I don't think my definition would approach. Let me just say how I would do it. Take your best, your favorite template. Anything you want. It could be a square, it could be a little disc, a circle, and you allow yourself to change, to zoom in and out with it and now you take many of those templates—you call those templates—whatever template you choose. You call that a unit, and you try to cover that object with those templates. Now the covering is not exact because it is overlap and under—it's just terrible, but you do it and you make a count of how many of your units you needed to do it. And now zoom—make it smaller, and cover by exactly the same, and make the appropriate computation and continue adding the item until you approach the actual area of the object. That's one way of doing it and I like that way, but that doesn't answer your question what is area.

Audience: No, and it's not unlike saying, “How would you define the square root of two?” But again, as a limit, it's the same kind of thing.

Mazur: Exactly.

Audience: I don't want to perturb the thread of this, so if you reject questions okay. It's kind of relating to an emphasis on the process, which you seem to enjoy more, and body. Many years ago when I was a math major, undergraduate, so don't ask any hard questions, I was amazed at the following phenomenon. I'd be working my butt off all night trying to get the answer, trying to get a proof—couldn't do it. I got up in the morning and it came like that! I guess it's related to body. The same process occurred many years later when I became a computer programmer. I would work all night on something and could not see it. Go away and step outside the building and it would come again. Is this an imaginative leap? Is it similar to Pascal—Pascal the mathematician—in his night of fire when he made the leap to faith? Why can I work all night on something and go to sleep and then the next morning I see the proof?

Brann: But you couldn't wake up the next morning and get it without having worked all night. We have a secret workshop that keeps going.

Audience: That brings to mind how much of what we're doing with mathematics is related to consciousnesses. Are we limited by our experiences? And if our brains are processing these operations without our conscious experience doing it, doesn't that beg an interesting question which is why are brains doing this? What is it about the organization of a human organic brain that seems to be able to do this without even the subject involved?

Brann: I have a colleague who a couple of days ago told me that there is a line in Plato's *Republic*—I stayed up far too late last night trying to find it. I don't think it's there, but it's wonderful, whether it's there or not. He claims it says that mathematicians tell wide-awake dreams that they had when asleep, that that's what mathematics is. You like that?

Mazur: I like that, yeah.

Brann: I don't know what it means.

Audience: I wanted to ask you too what extent has mathematics taken psychoanalysis as a fitting subject to explore in order to make some systematic effort to develop some molds, taking into account the fact that the average mathematician would probably not know very much about psychoanalytic phenomena or the formation of the psyche, the id, the ego, processes like transference. But nevertheless, since the human mind and personality operate according to definite rules that we've clinically validated, to what extent has mathematics seen fit to attempt to systematize—

Mazur: Oh, I think it has been systematized.

Audience: Can you point to the—

Mazur: **Well with Barbara's addendum**, let us say mathematics is based in language, and to the extent that one talks about psychoanalysis one uses language.

Brann: I think Freud's work on negation is much to the point. Does that ring a bell? He wrote that famous little article on negation, which essentially explained what it means to have, among other things, negative numbers, negativity of any sort. I think that's the case. Does anyone remember that article?

Audience: There's a line of mathematics which is more for predictability. In other words, the patient concentrated on certain aggressive feelings toward its mother, and one would be able to make a prediction as to what degree this would appear with certain frequency.

Mazur: Well, it hasn't been done of course.

Audience: Why not?

Audience: Low research funding.

Mazur: All right. I think we might be coming to a—

Audience: I was going to give you a choice to return to why math fits the world or a move towards mandalas, geometric visualization, and the very esoteric tradition of using mathematical imagery as a spiritual and meditative practice.

Mazur: That's interesting. I see. So you're pulling me in two completely different directions. In other words, let's go back to the vote, which is whether mathematics is—yes?

Audience: I just wanted to add one thing to the vote, which was the option of why math fits the world because we invented it that way. The idea that it was collectively pruned as an idea that evolved over time with the constraint for that pruning being reality. So we evolved an idea to meet reality. You gave us the square, for example. You have a system and we hit the constraints of our system, so we invent something to then allow us to keep progressing, and the reason we invent it is because we hit a constraint. The constraint arriving from reality, so obviously we arrive at something that very neatly explains reality, but that's how it came to existence in the first place. The collective pruning of this idea over centuries, over millennia, was in response to the reality constraint.

Mazur: That's interesting, but that's one part of the vote. Let me organize the possibilities so far. One is yours, which was in a sense there is the structure of our mind and the structure of our mind sort of describes, constrains if you wish, what we call explanation. If we systematize the types of things that seem to have objective explanations for us, given the structure of our minds we produce mathematics. That's one way of framing it. The other is the Platonic way, which is we are simply describers of the architecture of the cosmos and we describe it very well and so therefore it fits. The third would be that we're very good analyzers of ourselves. This is almost the psychoanalytic one, that the minute we say something that claims to be causal or claims to be explanatory, we analyze that, and we formalize that into a kind of logical and then super-logical structure which produces mathematics. That's in the direction of what is sometimes called the formalists.

Audience: How is that different from the first choice?

Mazur: The difference between that and one is partly cultural. It has nothing to do with the way in which we have to explain things, it's just kind of a cultural build-up. Those are the three standard ways that people talk about. I like the idea of voting because sometimes if you look at a piece of mathematics you say yes, it's culturally dependent. You look at another piece of mathematics and you say this is the architecture of the cosmos, and you look at a third and you say, this is the way in which my mind works. There's no way of coming definitively to any conclusion here, but there is a way of respecting the whole thing as something that is going to be with us as long as we do mathematics, this voting, so to speak.

Audience: Can I blur at least one distinction here? The distinction between the Platonic idea of the irrational number and the natural world in at least one respect and on one level as we know there are many examples in the natural world of things that seem to be aware of five—

Mazur: Aware of which?

Audience: The golden ratio. We see the nautilus shell that's divided in its spiral and we see, for example, the spirals of a sunflower and its seeds, the Fibonacci pattern in them. I just wanted to throw that out there so that we can blur some of these distinctions.

Mazur: That's actually right. The natural world pushes some very abstract mathematics on us—that's basically what you're saying. You just look at a sunflower seed and all of a sudden you're beginning to think of one plus the square root of five over two for some funny reason. I like that.

Audience: I don't think that's quite right. You think of rational numbers that are close to one plus the square root of five—

Mazur: Yeah, I know.

Audience: In the natural world the diagonal of the square is 1.41, something or other.

Mazur: And a two by four is two by three and a quarter, apparently.

Audience: That's something that I learned to my disdain once.

Mazur: Oh yes. That nail in the floor tells it.

Audience: But I think that the Platonic point is not recognizing the natural world. I think that mathematics recognizes the natural world as it ought to be but not as it is. It's some sort of a purely realized world—I shouldn't have said realized.

Mazur: You shouldn't have said "ought to be" either, by the way. You certainly feel that way sometimes.

Audience: Of these three ways of dealing with—what would St. Augustine—where would he have fit? If I recall, it's something like, if you ask me what time it is it's 8:15, but if you ask me what time is, I don't know. I believe he said something like that.

Mazur: I think he said, if you don't ask me what time is, I know it."

Brann: I have to tell you what Yogi Berra said. The best thing that was ever said about time that I've read is, someone asked him what time it was, and he said, "You mean now?"

Mazur: I think it's time to come to an end, but I want to thank—okay, one quick question.

Audience: Before you leave I wanted to ask you, can the language of personal relationships be transferred to mathematics? For example, are some of your best friends some of the mathematical ideas that you work with? Or do you have affection for certain mathematical objects that you create?

Mazur: Oh, but doesn't everybody have an affection for things that they are working on that they are obsessing on? Well, yes.

Audience: Just a natural human tendency.

Mazur: Certain numbers, like the number 37 and 691 are my favorites. I mean they really are, in some funny term.

Audience: Just a small comment. I'm an artist and I would like to get attention to, for example, this chair—how mathematical it is. It's a little bit simpler mathematics than, for example, shapes in this room. Probably this goes to the conclusion that we, our brains, are drawn to mathematics much more than, for example, real nature. When we go to real nature we can really see the straight lines, although here with your brain and mathematics—that we live in these structures, in all square buildings.

Mazur: Frank Gehry will not have it that way.

Audience: I'm going to try to phrase this properly. I'm way out of my league here. You discussed the natural world and then you discussed the human brain and you seem to make a distinction. How do you derive that distinction? Is it philosophical? Where does it come from?

Mazur: It's a good question. Okay, well let's have a spectrum of something which is not real, and that's your question—you're going to say that this whole spectrum is not real, but I'm going to make this spectrum. I call it the objectifying spectrum. At the very end of the objectifying spectrum is volcanoes and things like that, which I can talk about in measure. Getting closer to it would be my blood pressure, right. Then closer to that would be, maybe my—this is anybody's—brain tissues, which of course are objects. They're not us. Now the natural world is somewhere there and the psychological world is at the very far end of the objectifying spectrum, that is to say the un-objectifying end. I wouldn't want to make a clear distinction except workable ones, useable, sort of ad-hoc distinctions to be able to say yes, sometimes it's my thought and sometimes it's the natural world that I'm developing mathematics for and with, or by means of. I don't want to make the distinction very precise. Would you agree with that Eva or not?

Brann: To me it seems that there really is a fairly radical distinction, a real break, between beings that reflect and the beings that don't reflect, sticks and stones and universes. I have this feeling that aside from fictions and dreams the world doesn't think about me, but I think about the world

and that seems to me a very deep difference between thinking, being a subject, being an ego, a human being, and being a world. So I think that there really is a distinction. You can give any number, a large number of criteria, for what it is to be a thinking subject, which is the usual philosophical term for a thinking being and being in the world out there. Of course in that world there are other thinking subjects and a great issue is whether it's just human beings or possibly also animals. But that's not the main issue.

Audience: But I'm not sure that goes to the point of that which is natural. Is the human being natural? If so, and if not, what is he or she? And if so, are not the price of the composition natural?

Brann: That question seems to me complicated by the fact that there are two distinct meanings to natural—there may be more, but two that get involved in your question. One meaning of natural is given. Is it made or is it found? I think human beings aren't made—well, there might be religious aspects to this which I'm leaving out of it—but found. In that way human beings are natural and their thinking is natural. But the other meaning is that is natural which obeys laws, maybe deterministic, maybe probabilistic, but obeys laws and has to obey them. I don't think I need to do that. In other words, I think I'm to a certain degree freer than what we call nature is. So these are the two different ways of thinking of nature. Whether nature is deterministic or probabilistic, but law governed. I am not law governed. I don't need to explain to you why I'm not law governed.

Audience: Is evolution, which is a process we presumably come out of—is evolution normal?

Brann: There is a huge debate about that, whether evolution is a theory of law or of observation. It can't predict so it doesn't have the characteristics of physics. That gets us into a really interesting question.

Mazur: Okay, that's a really good way to end our discussion. Thank you all.